

Spin decoherence due to Fluctuating Fields

Piotr Szańkowski,^{1,2} M. Trippenbach,¹ and Y. B. Band²

¹*Institute of Theoretical Physics, University of Warsaw, ul. Hoża 69, PL-00-681 Warszawa, Poland*

²*Department of Chemistry, Department of Physics and Department of Electro-Optics,
and the Ilse Katz Center for Nano-Science, Ben-Gurion University, Beer-Sheva 84105, Israel*

The dynamics of a spin in the presence of a deterministic and a fluctuating magnetic field is solved for analytically to obtain the averaged value of the spin as a function of time for various kinds of fluctuations (noise). Specifically, analytic results are obtained for the time dependence of the expectation value of the spin, averaged over fluctuations, for Gaussian white noise and Gaussian colored noise, as well as non-Gaussian telegraph noise. Fluctuations cause the decay of the average spin vector (decoherence). For noise with finite temporal correlation time, a deterministic component of the field can suppress decoherence of the spin component along the field. Hence, decoherence can be manipulated by controlling the deterministic magnetic field. A simple universal physical picture emerges which explains the mechanism of the suppression of the decay.

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I. INTRODUCTION

A quantum system is almost invariably coupled to an environment that degrades its coherence. Decoherence of quantum systems is a fundamental problem, with implications across all branches of physics. The temporal evolution of a spin system, in the presence of coupling to an environment is perhaps the simplest phenomenon involving decoherence. For example, a particle of spin 1/2 [which can be mapped onto any two-level system (TLS) [1], and *vice versa*] has served as a useful paradigm [2–6]. Specific examples include the decoherence of a qubit in a quantum-computer or quantum-information processor [7], decoherence of an electron spin in ESR or a nuclear spin in NMR [8], spin dynamics in a magnetometer [9], accuracy and stability limitations of atomic clocks due to interactions with an environment [10], decoherence of two-level quantum dot systems [11, 13], and nitrogen vacancy centers in diamond [14, 16].

The main goal of this manuscript is to discuss the decoherence of a spin system caused by an environment. Under certain assumptions, which we discuss in the following section, the interaction of a spin with an environment can be represented by evolving the spin in an effective magnetic field, $\vec{B}^{(\text{eff})} = \vec{B}_0 + \vec{B}_E(t)$. Here \vec{B}_0 is a deterministic magnetic field (which can be time-dependent) that can be used to exert control over the spin. The magnetic field $\vec{B}_E(t)$ models the influence of an environment on the system, and is represented by a vector stochastic process $\vec{b}(t)$. Averaging over fluctuations corresponds to tracing out the environmental degrees of freedom. This yields a reduced, nonunitary dynamics wherein the averaged spin decoheres in time.

The physical properties of the environment determine the statistical properties of $\vec{B}_E(t)$, which in turn determine the type of stochastic process $\vec{b}(t)$. Here we consider several types of stochastic fluctuations. A prototype model for fluctuations is Gaussian white noise [18, 19], wherein the random process has vanishing correlation time. As a consequence, the system driven by it evolves without *memory*, since it has no way to probe the past. A more general type of noise is Gaussian colored noise [20, 21], wherein the random process has a finite correlation time, hence, the system will have memory. A particular case of colored noise that is *not* Gaussian, telegraph noise, has been studied in connection with quantum dot qubit systems [4–6]. A main result of our analysis is that, when the spin evolves with memory, it can be manipulated by applying a sufficiently strong deterministic field \vec{B}_0 to significantly suppress decoherence.

The outline of the this paper is as follows. Section II develops the equations of motion for a spin in the presence of a deterministic magnetic field and a stochastically fluctuating magnetic field. Section III describes evolution of the system with Gaussian colored noise fluctuations. In that section we present a simple, intuitive view of the nature of the decay of the averaged spin due to field fluctuations. We also show that this decay can be suppressed when the deterministic magnetic field is strong enough, and the correlation time is long enough (i.e., when the system has memory). Section IV develops the white noise limit of colored noise. Section VI considers telegraph noise, which is a specific form of non-Gaussian noise. Finally, a summary and a conclusion are presented in Sec. VII.

II. SPIN DYNAMICS IN A FLUCTUATING MAGNETIC FIELD

We consider fluctuations in an otherwise deterministic system that results from the application of a effective random field generated by the environment. The back-coupling of the spin to the source of the noise is not taken into account (Van Kampen [18] calls this *external noise*). Hence, the stochastic properties of the noise result only from the environment which is unaffected by the system. We assume that these properties can be measured or otherwise deduced. For each particular system, one should carefully check whether the back-action can indeed be neglected. Examples of systems for which back-action was neglected include: nitrogen-vacancy centers in diamond affected by magnetic moments of impurity spins [15, 16], electron spin in a quantum dot affected by nuclear magnetic moments that fluctuate due to crystal lattice vibrations [12, 13], and magnetic noise in atom chips caused by fluctuations of electron currents in wires that make up the atom chip [22].

Consider a particle of spin \vec{S} with magnetic moment $\vec{\mu} = \mu\vec{S}$ in the presence of a time-dependent field $\vec{B}(t)$. The dynamics is determined using a Zeeman-type Hamiltonian

$$H = -\vec{\mu} \cdot \vec{B}(t) = \vec{\Omega}(t) \cdot \vec{S}, \quad (1)$$

where \vec{S} is the vector of spin operators satisfying commutation relations $[S_i, S_j] = i\hbar\epsilon_{ijk}S_k$, $\vec{\Omega}(t) = -\mu\vec{B}(t)/\hbar$ is the Rabi frequency vector, $\vec{B}(t)$ is a magnetic field, $\mu = g\mu_0$ is the magnetic moment, g is the g -factor and μ_0 is the Bohr (or nuclear) magneton. The magnetic field $\vec{B}(t)$ is the sum of a deterministic field \vec{B}_0 , whose direction defines the z -axis, and a fluctuating field $\vec{b}(t)$, hence the Rabi frequency is given by

$$\vec{\Omega}(t) = \begin{pmatrix} 0 \\ 0 \\ \Omega_0(t) \end{pmatrix} + \begin{pmatrix} \omega_x(t) \\ \omega_y(t) \\ \omega_z(t) \end{pmatrix}, \quad (2)$$

where $\vec{\Omega}_0 = -\mu\vec{B}_0/\hbar$ and $\vec{\omega}(t) = -\mu\vec{b}(t)/\hbar$. We have taken the deterministic field to have only a component along the z -axis for simplicity.

In the Heisenberg representation, the evolution of the spin operators is given by the equations,

$$\frac{d}{dt}S_i^{(H)}(t) = \frac{i}{\hbar} [H(t), S_i^{(H)}(t)], \quad (3)$$

where the superscript (H) denotes the Heisenberg picture. In our case, the Hamiltonian contains a stochastic term, $\vec{\omega}(t)$:

$$H(t) = \Omega_0(t)S_z + \vec{\omega}(t) \cdot \vec{S}. \quad (4)$$

The commutator on the right hand side of the Heisenberg equation of motion, Eq. (3), is easy to evaluate, and leads to the vector form of the equation,

$$\frac{d}{dt}\vec{S}^{(H)}(t) = \{\Omega_0(t)\vec{e}_z + \vec{\omega}(t)\} \times \vec{S}^{(H)}(t), \quad (5)$$

and the initial condition is simply

$$\vec{S}^{(H)}(0) = \begin{pmatrix} S_x \\ S_y \\ S_z \end{pmatrix}. \quad (6)$$

For convenience, let us take the deterministic field Ω_0 to be constant in time. Then, the solution to Eq. (5) with the initial condition (6) can be written as a 3×3 evolution matrix, $\hat{U}(t, 0)$, acting on the initial vector, $\vec{S}^{(H)}(0)$. It is convenient to write the equations, $S_i^{(H)}(t) = \sum_{j=1}^3 U_{ij}(t, 0)S_j^{(H)}(0)$, in vector notation as $\vec{S}^{(H)}(t) = \hat{U}(t, 0)\vec{S}^{(H)}(0)$, where the hat $\hat{}$ indicates a 3×3 matrix. Substituting this form into the equations of motion yields

$$\frac{d}{dt}\hat{U}(t, 0) = \left(\Omega_0 \hat{\epsilon}^z + \sum_{k=1}^3 \omega_k(t) \hat{\epsilon}^k \right) \hat{U}(t, 0). \quad (7)$$

Here the matrix elements of $\hat{\epsilon}^k$ are defined as $\epsilon_{ij}^k \equiv \epsilon_{ikj}$, where ϵ_{ikj} is the Levi-Civita symbol. The matrices $\hat{\epsilon}^i$ satisfy commutation relations similar to those of angular momentum operators,

$$[\hat{\epsilon}^i, \hat{\epsilon}^j] = \epsilon_{ijk} \hat{\epsilon}^k. \quad (8)$$

Consequently, they are generators of rotations in a real, three-dimensional vector space. For example, $\hat{\varepsilon}^z$ generates rotations around z axis,

$$\hat{R}(\theta, z) \equiv \exp\{\theta \hat{\varepsilon}^z\} = \begin{pmatrix} \cos \theta & -\sin \theta & 0 \\ \sin \theta & \cos \theta & 0 \\ 0 & 0 & 1 \end{pmatrix}. \quad (9)$$

The evolution operator \hat{U} is a rotation operator that causes precession of the spin vector around the instantaneous rotation axis defined by the total field, and the instantaneous Rabi frequency vector is $\vec{\Omega}(t) = \vec{\Omega}_0 + \vec{\omega}(t)$. Hence \hat{U} is a stochastic matrix. The equation of motion of the spin does not depend upon the details of the representation of the spin, hence the solution to the dynamics for an arbitrary spin is also given by (7). The only indication of the spin representation, i.e., the dimension $2S + 1$ of the representation, is in the initial condition.

Experimental measurements of the spin will of necessity correspond to quantum expectation values and averages over the stochastic fluctuations of the magnetic field. Hence the quantity $\langle \vec{S}(t) \rangle$ corresponds to experimental measurements of the spin. Here, the symbol $\langle \rangle$ means quantum average, and the symbol $\overline{}$ means average over stochastic variables. Note that the average over fluctuations and the average over the initial quantum state factorizes in the following way: $\langle \vec{S}(t) \rangle = \overline{\hat{U}(t, 0)} \langle \vec{S}(0) \rangle$.

For spin $1/2$ (i.e., for a TLS), the average expectation value of the spin $\vec{S} = \hbar \vec{\sigma}/2$ fully determines the state of the system. This is because the density matrix ϱ can be expressed using the Pauli matrices $\vec{\sigma}$ which, together with identity matrix 1, form a basis in the space of 2×2 hermitian matrices. In the Schrödinger representation,

$$\overline{\varrho(t)} = \frac{1}{2} \left(1 + \text{Tr} \left\{ \overline{\vec{\sigma}^{(H)}(t)} \varrho(0) \right\} \cdot \vec{\sigma} \right) = \frac{1}{2} \left(1 + \overline{\langle \vec{\sigma}(t) \rangle} \cdot \vec{\sigma} \right). \quad (10)$$

and the expectation values of the spin at time t is given by vector parameter $\vec{\lambda}(t) \equiv \overline{\langle \vec{\sigma}(t) \rangle}$. Thus, at any time t , there are three parameters, $\lambda_i(t)$, $i = x, y, z$, which completely specify the density matrix. This does not mean that the dynamics of the density matrix for any $(2S + 1)$ -level system can be represented in terms of the dynamics of a particle with spin \mathbf{S} coupled to a magnetic field, since the number of parameters necessary to specify the density matrix larger than the three components of the spin [24].

III. GAUSSIAN COLORED NOISE

Initially, we consider a process that is Gaussian, isotropic and Markovian [18]. The field fluctuates in all three dimensions with independent components (isotropy) and statistical properties of each component is completely determined by the first two moments (Gaussian), the average and the correlation function. The stationary condition means that the correlation function depends on the time difference and the average is time independent. According to Doob's theorem [26], if the process is also Markovian, the correlation function is exponential and the average is zero, i.e.,

$$\overline{\omega_i(t)} = 0, \quad (11)$$

$$\overline{\omega_i(t) \omega_j(t')} = \kappa(t - t') \delta_{ij} = \omega_0^2 e^{-\frac{|t-t'|}{\tau_c}} \delta_{ij}. \quad (12)$$

Here the bar indicates average over the fluctuations, $\kappa(t - t')$ is the correlation function with correlation time τ_c , the average fluctuating Rabi frequency vanishes, $\vec{\omega} = 0$, and the variance of the Rabi frequency, ω_0^2 , specifies the strength of the fluctuations.

Several comments regarding Gaussian colored noise are in order. First, note that the Gaussian process can model a wide variety of environments due to the central limit theorem; if the environment consists of many individual and independent fluctuating elements, the central limit theorem substantiates the approximation of using a Gaussian process as the effective field affecting the spin. Secondly, if there is no back-action of the spin on the environment, and the environment is in a stationary state (e.g., is in equilibrium) then the process is stationary as well. Finally, the isotropy assumption regarding the random magnetic field produced by the environment is appropriate if no particular special direction can be associated with the environment.

To solve the equation of motion (7), first we eliminate the so called *drift term* due to the deterministic part of the magnetic field, $\Omega_0 \hat{\varepsilon}^z \hat{U}(t, 0)$, from the right hand side of the equation by transforming to the rotating frame. Then Eq. (7) takes the form

$$\frac{d}{dt} \hat{U}'(t, 0) = \left(\vec{\omega}(t) \cdot \vec{\varepsilon}'(t) \right) \hat{U}'(t, 0). \quad (13)$$

In this equation we introduced transformed matrices, denoted by a prime, $\vec{\varepsilon}'(t)$ and $\hat{U}'(t, 0)$, which are given by

$$(\varepsilon')_{ij}^k(t) \equiv \sum_{m,n} R_{im}(t\Omega_0, z) \varepsilon_{mn}^k R_{nj}(-t\Omega_0, z) = R_{kl}(t\Omega_0, z) \varepsilon_{ij}^l, \quad (14)$$

$$U'_{ij}(t_1, t_2) \equiv \sum_{m,n} R_{im}(t_1\Omega_0, z) U_{mn}(t_1, t_2) R_{nj}(-t_2\Omega_0, z), \quad (15)$$

where $\hat{R}(t\Omega_0, z) = \exp\{t\Omega_0\hat{\varepsilon}^z\}$ is a deterministic Euclidean rotation matrix that results in counterclockwise rotation around the z -axis by the angle $\Omega_0 t$ [see Eq. (9)]. Note that the scalar product $\vec{\omega}(t) \cdot \vec{\varepsilon}'(t)$ in Eq. (13) can be rewritten as

$$\vec{\omega}(t) \cdot \vec{\varepsilon}'(t) = \left(\hat{R}(-t\Omega_0, z) \vec{\omega}(t) \right) \cdot \vec{\varepsilon} \equiv \vec{\omega}'(t) \cdot \vec{\varepsilon}. \quad (16)$$

The time-dependent rotation matrix $\hat{R}(t\Omega_0, z)$ “mixes” field fluctuations in the x and y component. In the Lab frame, ω_x and ω_y are independent, however, because of the mixing, they are correlated in the rotating frame and the correlation functions are modified in the following way:

$$\overline{\omega'_x(t_1)\omega'_y(t_2)} = -\overline{\omega'_y(t_1)\omega'_x(t_2)} = \kappa(t_1 - t_2) \sin[\Omega_0(t_1 - t_2)], \quad (17)$$

$$\overline{\omega'_x(t_1)\omega'_x(t_2)} = \overline{\omega'_y(t_1)\omega'_y(t_2)} = \kappa(t_1 - t_2) \cos[\Omega_0(t_1 - t_2)], \quad (18)$$

$$\overline{\omega'_z(t_1)\omega'_i(t_2)} = \kappa(t_1 - t_2) \delta_{3i}. \quad (19)$$

The stochastic field in the rotating frame, $\vec{\omega}'(t)$, remains a Gaussian process, since it is a linear combination of Gaussian processes. Note that any time-independent rotation does not change the correlation functions of the isotropic Gaussian vector process $\vec{\omega}(t)$.

The formal solution to Eq. (13) is given in the terms of the time-ordered exponential function,

$$\hat{U}'(t, 0) = \mathcal{T} \exp \left\{ \sum_i \hat{\varepsilon}^i \int_0^t \omega'_i(\tau) d\tau \right\}, \quad (20)$$

defined by the power series, $\mathcal{T} \exp \int_0^t \hat{A}(\tau) d\tau \equiv \hat{1} + \int_0^t d\tau_1 \hat{A}(\tau_1) + \int_0^t d\tau_1 \int_0^{\tau_1} d\tau_2 \hat{A}(\tau_1) \hat{A}(\tau_2) + \dots$, where \mathcal{T} is the time-ordering operator [32]. The average value of \hat{U}' , $\overline{\hat{U}'(t, 0)}$, can be computed by noting that it has the form of a *moment generating functional* $\{M_X[\xi(t)]\}$ for matrix processes, $\hat{X}_i(t) \equiv \omega'_i(t)\hat{\varepsilon}^i$, $i = x, y, z$, where $\xi(t) = (\xi_x(t), \xi_y(t), \xi_z(t))$ is the vector of trial functions [38]. Then, the average can be “absorbed” inside the exponential function by expressing $M_X[\xi(t)]$ through a *cumulant generating functional* $\{K_X[\xi(t)]\}$ in terms of *cumulant averages* denoted here by a double bar (see Ref. [25] for a discussion of the cumulant expansion methods used here):

$$\begin{aligned} \overline{\hat{U}'(t, 0)} &= \overline{\mathcal{T} \exp \left\{ \sum_i \hat{\varepsilon}^i \int_0^t \omega'_i(\tau) d\tau \right\}} = M_X[\xi(t)] \Big|_{\substack{\xi(t) \equiv 1 \\ \hat{X}_i \equiv \omega'_i \hat{\varepsilon}^i}} \equiv \\ &\equiv \mathcal{T} \exp \{K_{\omega\hat{\varepsilon}}[1]\} = \mathcal{T} \exp \left\{ \overline{\mathcal{T} \exp \left(\sum_i \hat{\varepsilon}^i \int_0^t \omega'_i(\tau) d\tau \right)} - \hat{1} \right\} = \\ &= \mathcal{T} \exp \left\{ \sum_{i,j} \hat{\varepsilon}^i \hat{\varepsilon}^j \int_0^t d\tau_1 \int_0^{\tau_1} d\tau_2 \overline{\omega'_i(\tau_1) \omega'_j(\tau_2)} \right\}. \end{aligned} \quad (21)$$

In the last step, we took advantage of the fact that all cumulants beyond the second vanish for Gaussian processes, and $\overline{\omega'_i(t)\omega'_j(t')} = \overline{\omega'_i(t)\omega'_j(t')}$ if the average vanishes.

Substituting the correlation functions given by Eqs. (17), (18) and (19), we find the following expression for the cumulant generating functional K evaluated at $\xi(t) \equiv 1$,

$$K_{\omega'\hat{\varepsilon}}[1] = (\hat{\varepsilon}^z)^2 \int_0^t d\tau \gamma(\tau) + ((\hat{\varepsilon}^x)^2 + (\hat{\varepsilon}^y)^2) \int_0^t d\tau \gamma_+(\tau) + (\hat{\varepsilon}^x \hat{\varepsilon}^y - \hat{\varepsilon}^y \hat{\varepsilon}^x) \int_0^t d\tau \gamma_-(\tau), \quad (22)$$

where the rates γ are given by integrals of the correlation functions:

$$\gamma(t) = \int_0^t d\tau \overline{\omega'_z(\tau)\omega'_z(0)} = \int_0^t d\tau \kappa(\tau), \quad (23)$$

$$\gamma_+(t) = \int_0^t d\tau \overline{\omega'_x(\tau)\omega'_x(0)} = \int_0^t d\tau \overline{\omega'_y(\tau)\omega'_y(0)} = \int_0^t d\tau \kappa(\tau) \cos(\Omega_0\tau), \quad (24)$$

$$\gamma_-(t) = \int_0^t d\tau \overline{\omega'_x(\tau)\omega'_y(0)} = - \int_0^t d\tau \overline{\omega'_y(\tau)\omega'_x(0)} = \int_0^t d\tau \kappa(\tau) \sin(\Omega_0\tau). \quad (25)$$

K can be further simplified if we take into account properties of the rotation generators $\hat{\varepsilon}^i$. From the commutation relations (8), the commutator in the third term of Eq. (22) is equal to $\hat{\varepsilon}^z$. It is easy to verify that the sum of squares of $\hat{\varepsilon}^x$ and $\hat{\varepsilon}^y$ in the second term of K can be written as $\hat{\varepsilon}^{x^2} + \hat{\varepsilon}^{y^2} = -(2\hat{1} + \hat{\varepsilon}^2)$, hence, the parts of Eq. (22) commute with each other. Thus, the average evolution matrix, which is equal to the exponential of K , factorizes into three parts,

$$\begin{aligned} \overline{\hat{U}'(t,0)} &= \mathcal{T} \exp\{K_{\omega'\hat{\varepsilon}}[1]\} = \exp\left\{\hat{\varepsilon}^z \int_0^t d\tau \gamma(\tau) - (2\hat{1} + \hat{\varepsilon}^z{}^2) \int_0^t d\tau \gamma_+(\tau) + \hat{\varepsilon}^z \int_0^t d\tau \gamma_-(\tau)\right\} = \\ &= \underbrace{\begin{pmatrix} e^{-\Gamma(t)} & 0 & 0 \\ 0 & e^{-\Gamma(t)} & 0 \\ 0 & 0 & 1 \end{pmatrix}}_{\text{decay caused by the fluctuations in } z} \underbrace{\begin{pmatrix} e^{-\Gamma_+(t)} & 0 & 0 \\ 0 & e^{-\Gamma_+(t)} & 0 \\ 0 & 0 & e^{-2\Gamma_+(t)} \end{pmatrix}}_{\text{decay caused by the fluctuations in } xy \text{ plane}} \underbrace{\begin{pmatrix} \cos \Gamma_-(t) & -\sin \Gamma_-(t) & 0 \\ \sin \Gamma_-(t) & \cos \Gamma_-(t) & 0 \\ 0 & 0 & 1 \end{pmatrix}}_{\text{precession frequency modification due to fluctuations in } xy \text{ plane}}, \end{aligned} \quad (26)$$

where

$$\Gamma_+(t) = \int_0^t d\tau \gamma_+(\tau) = \left(\frac{\omega_0^2 \tau_c^2}{1 + \Omega_0^2 \tau_c^2}\right) \left[\frac{t}{\tau_c} + \left(\frac{1 - \Omega_0^2 \tau_c^2}{1 + \Omega_0^2 \tau_c^2}\right) \left(e^{-\frac{t}{\tau_c}} \cos \Omega_0 t - 1\right) - \left(\frac{2\Omega_0 \tau_c}{1 + \Omega_0^2 \tau_c^2}\right) e^{-\frac{t}{\tau_c}} \sin \Omega_0 t\right], \quad (27)$$

$$\Gamma_-(t) = \int_0^t d\tau \gamma_-(\tau) = \left(\frac{\omega_0^2 \tau_c^2}{1 + \Omega_0^2 \tau_c^2}\right) \left[\Omega_0 t + \left(\frac{1 - \Omega_0^2 \tau_c^2}{1 + \Omega_0^2 \tau_c^2}\right) e^{-\frac{t}{\tau_c}} \sin \Omega_0 t + \left(\frac{2\Omega_0 \tau_c}{1 + \Omega_0^2 \tau_c^2}\right) \left(e^{-\frac{t}{\tau_c}} \cos \Omega_0 t - 1\right)\right], \quad (28)$$

$$\Gamma(t) = \int_0^t d\tau \gamma(\tau) = \omega_0^2 \tau_c \left[t - \tau_c \left(1 - e^{-t/\tau_c}\right)\right]. \quad (29)$$

The first term on the right hand side of Eq. (26) describes the decay caused by the fluctuations in the direction parallel to the applied field $\vec{\Omega}_0 = \Omega_0 \hat{e}_z$. These fluctuations affect only spin components perpendicular to the z axis, and the decay rate $\Gamma(t)$ does not depend on the intensity of the constant field. The second term originates from the fluctuations in the xy plane, which is perpendicular to the applied field. Fluctuations in the plane couple to all the components of the spin and cause their decay. The key feature is that the rate of decay due to these fluctuations is dependent on Ω_0 . Finally, the third term, also related to the fluctuations in the xy plane, describes the modification of the precession frequency of the system caused by the noise.

The averaged expectation values of the spin are given by:

$$\overline{\langle S_x(t) \rangle} = e^{-[\Gamma(t) + \Gamma_+(t)]} \left\{ \cos [\Omega_0 t + \Gamma_-(t)] \langle S_x \rangle - \sin [\Omega_0 t + \Gamma_-(t)] \langle S_y \rangle \right\}, \quad (30)$$

$$\overline{\langle S_y(t) \rangle} = e^{-[\Gamma(t) + \Gamma_+(t)]} \left\{ \sin [\Omega_0 t + \Gamma_-(t)] \langle S_x \rangle + \cos [\Omega_0 t + \Gamma_-(t)] \langle S_y \rangle \right\}, \quad (31)$$

$$\overline{\langle S_z(t) \rangle} = e^{-2\Gamma_+(t)} \langle S_z \rangle. \quad (32)$$

In the limit $\Omega_0 \rightarrow 0$, we obtain $\Gamma_+(t) \rightarrow \Gamma(t)$ and $\Gamma_-(t) \rightarrow 0$, hence Eqs. (30)-(32) reduce to

$$\overline{\langle \vec{S}(t) \rangle} \xrightarrow{\Omega_0 \rightarrow 0} e^{-2\Gamma(t)} \langle \vec{S} \rangle. \quad (33)$$

Substituting Eq. (29) into (33) yields the analytic form (33) of the isotropic decay of the average spin, regardless of the initial state of the spin, as can be seen in Fig. 1. For times $t \gg \tau_c$, the decay is simply exponential with the rate of decay given by $\Gamma(t) \approx \omega_0^2 \tau_c t$. For times $t \ll \tau_c$, the rate of decay is $\Gamma(t) \approx \omega_0^2 t^2 / 2$. In the absence of an external field, the fluctuations in all three directions remain independent and the average evolution matrix decomposes in the

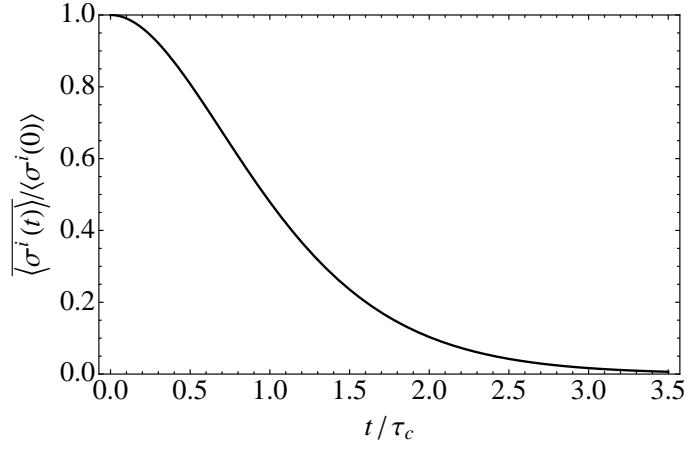


FIG. 1: The expectation value of the spin, $\overline{\langle \vec{S}(t) \rangle}$, subject to Gaussian colored noise, versus time for $\Omega_0 = 0$ and $\omega_0\tau_c = 1$. The analytic formula for $\overline{\langle \vec{S}(t) \rangle}$ is given in Eq. (33).

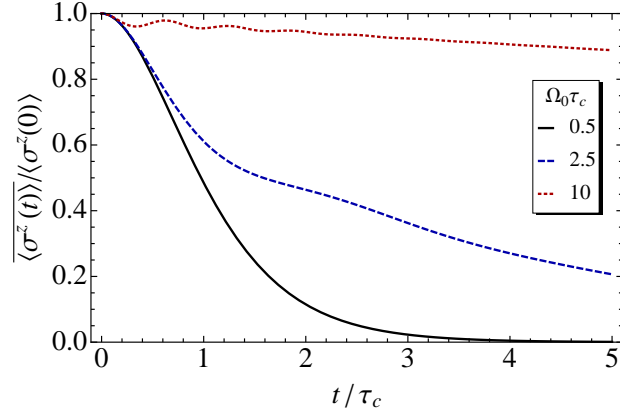


FIG. 2: $\overline{\langle S_z(t) \rangle}$ subject to Gaussian colored noise versus time for nonvanishing constant Ω_0 (see legend) and $\omega_0\tau_c = 1$. Equation (32) gives the analytic formula for $\overline{\langle S_z(t) \rangle}$.

following way:

$$\overline{\vec{U}(t, 0)}\Big|_{\Omega_0=0} = \underbrace{\begin{pmatrix} e^{-\Gamma(t)} & 0 & 0 \\ 0 & e^{-\Gamma(t)} & 0 \\ 0 & 0 & 1 \end{pmatrix}}_{\text{fluctuations in } z} \underbrace{\begin{pmatrix} e^{-\Gamma(t)} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & e^{-\Gamma(t)} \end{pmatrix}}_{\text{fluctuations in } y} \underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 0 & e^{-\Gamma(t)} & 0 \\ 0 & 0 & e^{-\Gamma(t)} \end{pmatrix}}_{\text{fluctuations in } x}. \quad (34)$$

Here the first part comes from fluctuations in z and causes the decay of the x and y components of the spin, the second part comes from fluctuations in y and causes the decay of the perpendicular components, and the third part originates from fluctuations in x direction causes the decay of the y and z components of the spin.

Now let us consider finite Ω_0 . The dependence of $\overline{\langle S_z(t) \rangle}$, the component parallel to the external field $\vec{\Omega}_0$, versus time for three values of dimensional parameter $\Omega_0\tau_c$ is plotted in Fig. 2. As we discussed previously, z component of the average spin decays due to fluctuations in the xy plane, which are affected by the presence of the constant field. For $\Omega_0\tau_c \ll 1$, the quantity $\overline{\langle S_z(t) \rangle}$ is clearly very similar to the results plotted in Fig. 1, since $\Gamma_+(t) \rightarrow \Gamma(t)$. As $\Omega_0\tau_c$ increases, the decay of $\overline{\langle S_z(t) \rangle}$ is significantly slowed, and $\Gamma_+(t) \rightarrow 0$ in the limit of $\Omega_0\tau_c \rightarrow \infty$, i.e., the decay is fully suppressed.

Figure 3 plots $\overline{\langle S_x(t) \rangle}$ versus time for the case when $\langle S_y(0) \rangle = 0$ and Fig. 4 presents the time dependence of the length of the vector $\overline{\langle \vec{S}_\perp(t) \rangle} = (\overline{\langle S_x(t) \rangle}, \overline{\langle S_y(t) \rangle}, 0)$. The decay of the components of the average spin perpendicular to the constant field $\vec{\Omega}_0$ is not only induced by the fluctuations in the xy plane, but also by those in the z direction [see Eq. (26)]. We have previously seen that the effects of the fluctuations in x and y are inhibited if $\Omega_0\tau_c$ is large enough.

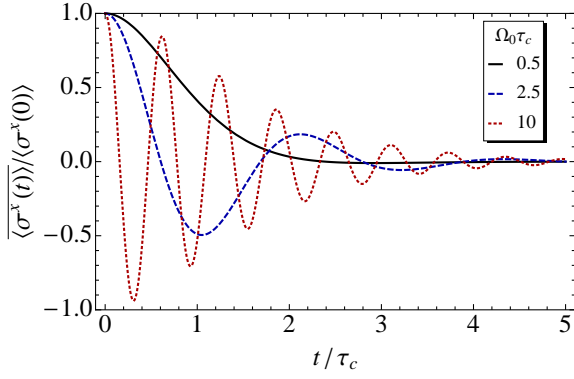


FIG. 3: The average value $\langle S_x \rangle$, when the spin is subject to Gaussian colored noise for nonvanishing constant Ω_0 (see legend) and $\omega_0 \tau_c = 1$. Equation (30) together with initial condition $\langle S_y(0) \rangle = 0$ yield the analytic formula for the function plotted. The average value $\langle S_y \rangle$ for initial condition $\langle S_x(0) \rangle = 0$ is identical to the results shown here.

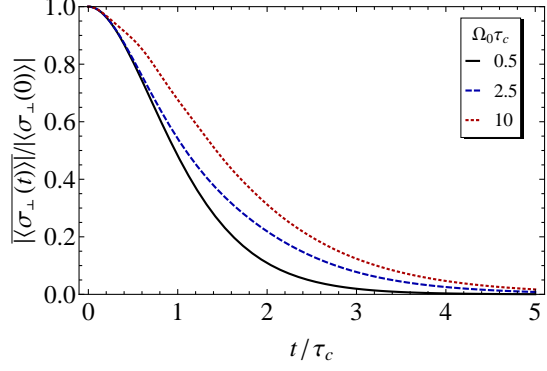


FIG. 4: Average value of $|\langle \sigma_\perp(t) \rangle| = \sqrt{\langle S_x(t) \rangle^2 + \langle S_y(t) \rangle^2}$ subject to Gaussian colored noise for nonvanishing constant Ω_0 (see legend) and $\omega_0 \tau_c = 1$. Equations (30) and (31) yield the analytic formula for the function plotted.

However, the influence of the fluctuations in the direction parallel to applied field is unperturbed by the presence of Ω_0 . As long as the z component of the field is fluctuating the perpendicular component of the average spin will decay to zero for any value of $\Omega_0 \tau_c$ as $t / \tau_c \rightarrow \infty$. For small $\Omega_0 \tau_c$ the decay is, again, similar to the isotropic case shown in Fig. (2), but when $\Omega_0 \tau_c$ increases, the decay is modulated by oscillations which result from the precession of the spin about the external field $\vec{\Omega}_0$ with frequency modified by the fluctuations. For $t \gg \tau_c$, the oscillation frequency is given by $\Omega_0 \left(1 + \frac{\omega_0^2 \tau_c^2}{1 + \Omega_0^2 \tau_c^2}\right)$ [see Eq. (28)]. When $\Omega_0 \tau_c \gg 1$ the frequency of precession tends to Ω_0 as the frequency modification due to fluctuations, $\Gamma_-(t)$ vanishes, in a similar manner as that of $\Gamma_+(t)$.

For spin 1/2 system (a TLS), the decay of $\langle S_x(t) \rangle$ and $\langle S_y(t) \rangle$ correspond to the decay of the coherence of the system, whereas the decay of $\langle S_z(t) \rangle$ corresponds to the vanishing of population imbalance. If the constant field is taken along x (as opposed to along z), decay of $\langle S_x(t) \rangle$ is suppressed, and the decoherence time $T_2 \sim \tau_c \Omega_0^2 / \omega_0^2$, which can be made arbitrarily large (suppressed decay) by increasing Ω_0 .

IV. THE WHITE NOISE LIMIT

The white noise limit of colored noise involves going to the regime of extremely rapid fluctuations of the field, i.e., vanishing correlation time τ_c . Unfortunately, simply taking $\tau_c \rightarrow 0$ does not yield the correct limit. If we put $\tau_c = 0$ in Eq. (29), we obtain that the decay rate of the average spin equals zero, and fluctuations do not affect the evolution of the spin. The proper limit can be obtained if the strength of fluctuations, as given by the variance of the field, $\omega(t)^2 = \omega_0^2$, is increased as the correlation time τ_c goes to zero, so that $\omega_0^2 \tau_c$ remains constant. Thus, we arrive at the limit when the correlation function of the colored noise becomes proportional to a Dirac delta function,

$$\overline{\omega_i(t) \omega_j(t')} = \omega_0^2 e^{-\frac{|t-t'|}{\tau_c}} \delta_{ij} = 2\omega_0^2 \tau_c \left(\frac{1}{2\tau_c} e^{-\frac{|t-t'|}{\tau_c}} \right) \delta_{ij} \xrightarrow[\tau_c \rightarrow 0]{\omega_0^2 \tau_c \rightarrow D} 2D \delta(t-t') \delta_{ij}. \quad (35)$$

In the limit of $\kappa(\tau) \rightarrow 2D \delta(\tau)$ the decay rates $\Gamma_+(t) = \Gamma(t) = 2Dt$ and $\Gamma_-(t) = 0$. The decay of the expectation value of the spins are then isotropic and are not affected by the presence of a constant field $\vec{\Omega}_0$, i.e.,

$$\langle \vec{S}(t) \rangle = \begin{pmatrix} e^{-\Gamma(t)-\Gamma_+(t)} \cos \Gamma_-(t) & -e^{-\Gamma(t)-\Gamma_+(t)} \sin \Gamma_-(t) & 0 \\ e^{-\Gamma(t)-\Gamma_+(t)} \sin \Gamma_-(t) & e^{-\Gamma(t)-\Gamma_+(t)} \cos \Gamma_-(t) & 0 \\ 0 & 0 & e^{-2\Gamma_+(t)} \end{pmatrix} \hat{R}(\Omega_0 t, z) \langle \vec{S} \rangle \rightarrow e^{-4Dt} \hat{R}(\Omega_0 t, z) \langle \vec{S} \rangle. \quad (36)$$

Thus, the spin decay is isotropic, purely exponential with decay rate $4D$, and is independent of Ω_0 . Moreover, the precession due to $\vec{\Omega}_0$ is unaffected by the fluctuations. In the white noise limit, the precession due to deterministic part of the field and the decoherence due to field fluctuations are completely decoupled.

V. MEMORY EFFECTS

In the expression for $\overline{\langle \vec{S}(t) \rangle}$ for the case of Gaussian colored noise, the decay rates of the spin depend on the history of the fluctuating field through the integrals of correlation functions. But the decay rate for the white noise case does not depend on history. Based on this observation, we say that spin system has *memory* if the fluctuating field driving it has finite correlation time. This definition is consistent with a rigorous notion of memory, since $\vec{S}(t)$, understood as a stochastic process, is Markovian [18] (i.e. memoryless) if and only if the noise driving it is white noise. Memory in the system has important consequences, because it allows control of spin decay, through the application of external field.

The role of memory can be understood as follows. Consider first the case of no constant field. Each realization of the fluctuating field $\vec{\omega}(t)$ determines corresponding realization of $\langle \vec{S}(t) \rangle = \hat{U}(t, 0) \langle \vec{S} \rangle$ via the equation of motion, where $\vec{\omega}(t)$ defines an instantaneous rotation axis for the spin vector. Hence, each realization draws a trajectory of the spin on a sphere of radius $|\langle \vec{S} \rangle|$. The evolution can be viewed in a simplified “stroboscopic” picture in which, within a correlation time τ_c , the axis $\vec{\omega}(t)$ does not change much, and subsequently, it jumps to a new random value within a range on the order of ω_0 . Eventually such a sequence of rotations around randomly selected axes “smears” the trajectories of the spin over the whole sphere. This results in the decay of the spin averaged over all realizations.

The situation is different when a constant field along, say, the z -axis is present. This becomes clear when we view the evolution in the reference frame rotating with frequency Ω_0 around the z -axis. Consider a particular realization of the fluctuating field $\vec{\omega}(t)$, and denote it in the rotating frame as $\vec{\omega}'(t)$. Since $\omega'_z(t) = \omega_z(t)$, the fluctuations in the direction parallel to the constant field are unchanged when shifted to the rotating frame. On the other hand, the projection of the fluctuating field on the plane perpendicular to z -axis, $\vec{\omega}_\perp(t) = (\omega_x(t), \omega_y(t), 0)$, is modified when viewed in the rotating frame. Within a time of order the correlation time τ_c , the projection $\vec{\omega}'_\perp(t)$ revolves around the z -axis with frequency Ω_0 , and if $\Omega_0 \tau_c \gg 1$, $\vec{\omega}'_\perp(t)$ is able to perform many revolutions between jumps. The effect of the jumps becomes negligible in comparison with the systematic rotation. Thus the stochastic character of $\vec{\omega}'_\perp(t)$ is lost, and the only contribution to the decay caused by fluctuations comes from $\omega_z(t)$. A similar effect provides stable motion of a spinning top, and for that reason we will refer to it as a *gyroscopic effect*. If $\Omega_0 \tau_c \ll 1$, then revolution in between jumps is negligible, so $\vec{\omega}'_\perp(t) \approx \vec{\omega}_\perp(t)$, and by necessity the decay induced by the fluctuations must be almost the same as if there was no constant field at all.

The suppression mechanism of the decay described above is inoperative in the white noise limit where the correlation time vanishes, so the fluctuations of the rotation axis are instantaneous. Hence, the decay caused by white noise cannot be affected by a constant field of finite intensity. Moreover, even if the strength of the fluctuations is small in comparison to Ω_0 , the effect of fluctuations cannot be neglected if the correlation time is too short.

Note that the important dimensionless parameter here is $\Omega_0 \tau_c$, and the conclusion of this paragraph remain valid even for non-Gaussian noise as long as it has nonvanishing correlation time. We verify this assessment in the following section.

VI. NON-GAUSSIAN PROCESS: TELEGRAPH NOISE

In this section we present an example of a non-Gaussian process, telegraph noise, i.e., dichotomic noise corresponding to a memoryless continuous-time stochastic process that jumps between two distinct values, sometimes called burst noise [18, 33]. We consider telegraph noise in one component of the field, $\omega_z(t)$. The Rabi frequency of the fluctuating field, $\omega_z(t)$, jumps randomly between $+\omega_0$ and $-\omega_0$, with a jump rate $w = (2\tau_c)^{-1}$. The average value and the correlation function of such a process is given by

$$\overline{\omega_z(t)} = 0, \quad \overline{\omega_z(t)\omega_z(t')} = \omega_0^2 e^{-2w|t-t'|} = \omega_0^2 e^{-\frac{|t-t'|}{\tau_c}}. \quad (37)$$

Because telegraph noise is non-Gaussian, the correlation function $\kappa(t-t') = \omega_0^2 e^{-\frac{|t-t'|}{\tau_c}}$ does not contain all the information regarding the process.

Telegraph noise can model the effect of a spin 1/2 impurity atom near the spin [3, 5, 6]. The impurity spin has probabilities p_+ and p_- for being in the spin-up and spin-down state, and a hopping rate for transferring from one spin state to the other. Neglecting the back-action of the magnetic moment on the impurity, this model reduces to telegraph noise.

Our goal is to find the average evolution matrix $\overline{\hat{U}(t, 0)}$ for the telegraph noise case. The technique we have used previously to find the average does not work in the case of telegraph noise, because unlike with Gaussian processes, the closed form of the cumulant generating functional is not known. Fortunately, it is still possible to find an exact, analytical formula for $\hat{U}(t, 0)$ if the fluctuations are parallel to the constant field. One way of solving for $\hat{U}(t, 0)$ is

to follow Ref. [34]. We can write the average evolution operator as $\overline{\hat{U}(t,0)} = \overline{\hat{U}(t,0)}_+ + \overline{\hat{U}(t,0)}_-$, where $\overline{\hat{U}(t,0)}_{\pm}$ are the averages conditional on $\omega_z(t)$ being equal to $\pm\omega_0$. These averages vary in time because of equation of motion (7), and because $\omega_z(t)$ jumps. Thus,

$$\frac{d}{dt}\overline{\hat{U}(t,0)}_+ = (\Omega_0 + \omega_0)\varepsilon^z\overline{\hat{U}(t,0)}_+ - \frac{1}{2\tau_c}\overline{\hat{U}(t,0)}_+ + \frac{1}{2\tau_c}\overline{\hat{U}(t,0)}_-, \quad (38)$$

$$\frac{d}{dt}\overline{\hat{U}(t,0)}_- = (\Omega_0 - \omega_0)\varepsilon^z\overline{\hat{U}(t,0)}_- - \frac{1}{2\tau_c}\overline{\hat{U}(t,0)}_- + \frac{1}{2\tau_c}\overline{\hat{U}(t,0)}_+. \quad (39)$$

Adding and subtracting these equations, we obtain

$$\frac{d}{dt}\overline{\hat{U}(t,0)} = \Omega_0\varepsilon^z\overline{\hat{U}(t,0)} + \omega_0\varepsilon^z\overline{\Delta\hat{U}(t,0)}, \quad (40)$$

$$\frac{d}{dt}\overline{\Delta\hat{U}(t,0)} = \Omega_0\varepsilon^z\overline{\Delta\hat{U}(t,0)} + \omega_0\varepsilon^z\overline{\hat{U}(t,0)} - \frac{1}{\tau_c}\overline{\Delta\hat{U}(t,0)}, \quad (41)$$

where $\overline{\Delta\hat{U}(t,0)} = \overline{\hat{U}(t,0)}_+ - \overline{\hat{U}(t,0)}_-$. The next step is to eliminate $\overline{\Delta\hat{U}(t,0)}$ from Eq. (40) by substituting the solution to Eq. (41), with the initial condition $\overline{\hat{U}(0,0)}_{\pm} = \hat{1}/2$. This yields a closed equation for average evolution matrix:

$$\frac{d}{dt}\overline{\hat{U}(t,0)} = \Omega_0\varepsilon^z\overline{\hat{U}(t,0)} + \omega_0^2 \int_0^t dt' e^{-t'/\tau_c} (\varepsilon^z)^2 \overline{\hat{U}(t-t',0)}. \quad (42)$$

Note the particular form of the “memory integral” in the equation (42), where the dynamical variable is evaluated at times previous to t . In this case the system clearly has memory and it remembers not only the history of the noise but also that of its own. It is easy to check that the solution to this equation which satisfies the initial condition $\overline{\hat{U}(0,0)} = \hat{1}$ is given by

$$\overline{\hat{U}(t,0)} = e^{-t/2\tau_c} \left[\cosh\left(\hat{\gamma}\frac{t}{2\tau_c}\right) + \hat{\gamma}^{-1} \sinh\left(\hat{\gamma}\frac{t}{2\tau_c}\right) \right] \hat{R}(\Omega_0 t, z), \quad (43)$$

where

$$\hat{\gamma} = \begin{pmatrix} \sqrt{1-4\omega_0^2\tau_c^2} & 0 & 0 \\ 0 & \sqrt{1-4\omega_0^2\tau_c^2} & 0 \\ 0 & 0 & 1 \end{pmatrix}. \quad (44)$$

The average expectation values of the spin are obtained by applying the average evolution matrix (43) to the initial vector of spin operators, $\overline{\langle \vec{S}(t) \rangle} = \overline{\hat{U}(t,0)} \langle \vec{S}(0) \rangle$, which yields,

$$\overline{\langle S_x(t) \rangle} = e^{-\frac{t}{2\tau_c}} \left[\cosh\left(\sqrt{1-4\omega_0^2\tau_c^2}\frac{t}{2\tau_c}\right) + \frac{\sinh\left(\sqrt{1-4\omega_0^2\tau_c^2}\frac{t}{2\tau_c}\right)}{\sqrt{1-4\omega_0^2\tau_c^2}} \right] [\cos(\Omega_0 t) \langle S_x \rangle - \sin(\Omega_0 t) \langle S_y \rangle], \quad (45)$$

$$\overline{\langle S_y(t) \rangle} = e^{-\frac{t}{2\tau_c}} \left[\cosh\left(\sqrt{1-4\omega_0^2\tau_c^2}\frac{t}{2\tau_c}\right) + \frac{\sinh\left(\sqrt{1-4\omega_0^2\tau_c^2}\frac{t}{2\tau_c}\right)}{\sqrt{1-4\omega_0^2\tau_c^2}} \right] [\cos(\Omega_0 t) \langle S_y \rangle + \sin(\Omega_0 t) \langle S_x \rangle], \quad (46)$$

$$\overline{\langle S_z(t) \rangle} = \langle S_z \rangle. \quad (47)$$

Neither constant field Ω_0 , nor field fluctuations $\omega_z(t)$ couple to the z component of the spin, hence it remains constant throughout the evolution. The effects of the presence of Ω_0 and $\omega_z(t)$ on the components of the spin in xy plane are simply superposed: constant field causes trivial precession around z axis, while the noise makes the x and y spin components decay on average. Moreover, if the constant field and the fluctuating field are parallel, the gyroscopic effect can not be observed. The only memory effect present in this case is a quadratic behavior in t of the decay rate at $t \ll \tau_c$. Figure 5 plots $\overline{\langle S_x(t) \rangle}$ versus time for telegraph noise in $\omega_z(t)$, for the case of no static external magnetic present. As $\omega_0\tau_c$ increases, $\overline{\langle S_x(t) \rangle}$ decays more quickly, but for $\omega_0\tau_c > 1/2$, the square root $\sqrt{1-4\omega_0^2\tau_c^2}$ becomes imaginary and the solution changes form from pure decay to decay with oscillations.

It is of interest to see what happens when noise is present in the component of the field perpendicular to the constant field Ω_0 , e.g., in $\omega_x(t)$. Unfortunately, in this case, it is not possible to obtain an analytical formula for the average evolution matrix. According to the previous discussion, one would expect that, since telegraph noise is colored noise (the correlation time τ_c is finite), the gyroscopic effect should work in a fashion similar to the Gaussian colored noise example. This is indeed the case. Numerical solutions of $\overline{\langle S_z(t) \rangle}$ versus time are plotted in Fig. 6 and suppression of the decay is more efficient with increasing $\Omega_0\tau_c$.

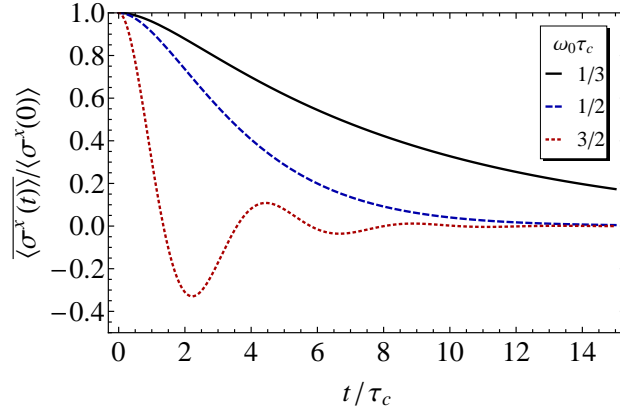


FIG. 5: $\overline{\langle S_x(t) \rangle}$ versus time for a spin subject to telegraph noise in only the z -component of the field, $\omega_z(t)$, and $\Omega_0 = 0$. $\overline{\langle S_x(t) \rangle}$ is identical. Nonvanishing values of Ω_0 simply cause precession of the spin about the z axis.

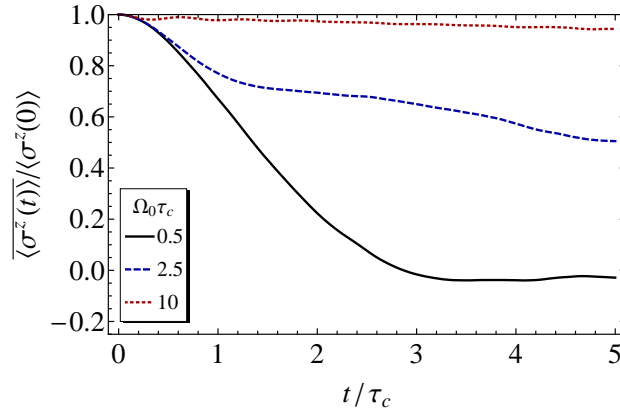


FIG. 6: $\overline{\langle S_z(t) \rangle}$ versus time for a spin subject to telegraph noise only in the x direction, with $\omega_0 \tau_c = 1$, perpendicular to the constant field Ω_0 which is in the z direction. As the parameter $\Omega_0 \tau_c$ is increased (see legend), suppression of the decay of the spin increases.

VII. SUMMARY AND CONCLUSION

We introduced a stochastic model for spin in the presence of a deterministic magnetic field and a fluctuating magnetic field (noise), and we derived the stochastic equations of motion for the spin vector $\langle \vec{S}(t) \rangle$. The environmental degrees of freedom, represented by the fluctuations of the field, are eliminated by taking the average over the fluctuations. We made the *external noise* assumption, wherein no back-action of the system on the environment is present. The most pronounced consequence of this assumption is that the system does not go into equilibrium with a thermal environment, but instead goes to the most democratic density matrix, hence the system tends towards steady state of the completely mixed state, and $\overline{\langle \vec{S}(t) \rangle} \rightarrow 0$ as $t \rightarrow \infty$. This is in contradistinction to the case of noise experienced by a system in contact with a bath, wherein the bath affects the system *and* the vice versa. The lack of mutual interaction (back-action) means that the fluctuation-dissipation theorem [17] cannot be applied, and a thermal equilibrium state of the system cannot be achieved at large times. If back-action on the bath due to the system is present, the source of the fluctuations and the dissipation is the same, they are connected through fluctuation-dissipation theorem, and thermal equilibrium of the system must result.

We explicitly considered Gaussian colored noise, Gaussian white noise and non-Gaussian telegraph noise. From our studies we conclude that if the system has memory, i.e., it has a finite correlation time, then the system can be manipulated by means of an external magnetic field, and the decoherence induced by the fluctuations can be significantly suppressed. In contradistinction, in the white noise limit, when the correlation function tends to a Dirac delta function, the decoherence cannot be controlled in this way. We introduced simple and intuitive considerations for why the suppression of decay can be achieved for any stochastic process as long as the correlation time is long

enough.

In future publications we will present a quantum master equation approach based upon the techniques developed in this work [23], wherein quantum master equations are derived and yield an equation for the density matrix for the spin. Before averaging over the random magnetic field, the equations of motion are stochastic because of the coupling of the stochastic magnetic field to the spin. Averaging over the stochastic processes can be calculated analytically to yield a deterministic equation of motion for the density matrix averaged over the stochastic magnetic field realizations. We will also discuss the effects of stochastic fields on two-qubit systems that are and are not entangled, to see how the correlation and entanglement of the qubits are affected by different kinds of noise [36]. Moreover, we will explicitly explore the effects of magnetic noise on magnetometers and atomic clocks [37].

Acknowledgments

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- [1] R. P. Feynman and F. L. Vernon, *Ann. Phys. (N.Y.)* **24**, 118 (1963).
 - [2] J. R. Klauder and P. W. Anderson, *Phys. Rev.* **125**, 912 (1962).
 - [3] T. Itakura and Y. Tokura, *Phys. Rev. B* **67**, 195320 (2003).
 - [4] J. Bergli, Y. M. Galperin, and B. L. Altshuler, *New J. Phys.* **11**, 025002 (2009).
 - [5] B. Cheng, Q. H. Wang, and R. Joynt, *Phys. Rev. A* **78**, 022313 (2008).
 - [6] A. Aharony, S. Gurvitz, O. Entin-Wohlman and S. Dattagupta *Phys. Rev. B* **82**, 245417 (2010).
 - [7] C. Bennett, *Physics Today* **48** (10), 24 (1995); D. P. DiVincenzo, *Science* **270**, 255 (1995); A. M. Steane, *Rep. Prog. Phys.* **61**, 117 (1998).
 - [8] A. Abragam, *Principles of Nuclear Magnetism*, (Oxford University Press, 1983).
 - [9] D. Budker and M. Romalis, *Nature Phys.* **3**, 227 (2007).
 - [10] J. Vanier and C. Audoin, *The Quantum Physics of Atomic Frequency Standards*, (Adam Hilger IOP Publishing Ltd., Bristol, 1989); P. Lemonde, *et al.*, “Cold-Atom Clocks on Earth and in Space”, in *Frequency Measurement and Control*, (Springer-Verlag, Berlin, 2001); *Topics Appl. Phys.* **79**, 131-153 (2001).
 - [11] D. Loss and D. P. DiVincenzo, *Phys. Rev. A* **57**, 120 (1998); T. Tanamoto, *ibid.* **61**, 022305 (2000).
 - [12] V.V. Dobrovitski *et al.*, *Phys. Rev. Lett.* **102**, 237601 (2009).
 - [13] J.M. Taylor *et al.*, *Phys. Rev. B* **76**, 035315 (2007).
 - [14] L. Childress *et al.*, *Science* **314**, 281 (2006); R. Hanson *et al.*, *Science* **320**, 352 (2008); J. R. Maze *et al.*, *Nature* **455**, 644 (2008).
 - [15] G. de Lange *et al.*, *Science* **330**, 60 (2010).
 - [16] A. Bermudez, F. Jelezko, M. B. Plenio, and A. Retzker, *Phys. Rev. Lett.* **107**, 150503 (2011).
 - [17] H. B. Callen and T. A. Welton, *Phys. Rev.* **83**, 34 (1951).
 - [18] N. G. Van Kampen, *Stochastic Processes in Physics and Chemistry*, (Elsevier Science, 1997).
 - [19] P. E. Kloeden and E. Platen, *Numerical Solution of Stochastic Differential Equations*, (Springer, 2011).
 - [20] M. C. Valsakumar, *J. of Stat. Phys.* **39**, 347 (1985).
 - [21] N. G. van Kampen, *J. of Stat. Phys.* **54**, 1289 (1989).
 - [22] V. Dikovsky, Y. Japha, C. Henkel and R. Folman, *Eur. Phys. J. D.* **35**, 87 (2005).
 - [23] P. Szańkowski, M. Trippenbach and Y. B. Band, “Master Equation Approach to Decoherence of Two-Level Systems due to Fluctuating Fields”, (to be published).
 - [24] The number of independent parameters needed to characterize the density matrix of a spin S system is $(2S + 1)^2 - 1$, so for $S = 1/2$ the three quantities $\langle S_i \rangle$ are sufficient, but for higher S , the number of parameters exceeds the number of expectation values of the spin components.
 - [25] R. Kubo, *J. Phys. Jap.* **17**, 1100 (1962).
 - [26] J. L. Doob, *Ann. Math. Statist.* **15**, 229 (1944).
 - [27] A. J. Leggett, *et al.*, *Rev. Mod. Phys.* **59**, 1 (1987).
 - [28] R. P. Feynman, *Phys. Rev.* **84**, 108 (1951).
 - [29] U. Fano, *Phys. Rev.* **96**, 869 (1954).
 - [30] U. Fano, *Rev. Mod. Phys.* **29**, 74 (1957).
 - [31] G. Lindblad, *Comm. Math. Phys.* **48**, 119 (1976).
 - [32] N. G. van Kampen, *Physica* **74**, 239 (1974).
 - [33] C. W. Gardiner, *Handbook of Stochastic Methods*, (Springer, 2004).

- [34] N. G. van Kampen *Physica* **70**, 222 (1973).
- [35] F. Bloch, *Phys. Rev.* **70**, 460 (1946).
- [36] P. Szańkowski, M. Trippenbach and Y. B. Band, “Entanglement Decoherence of Two Qubit Systems due to Fluctuating Fields”, (to be published).
- [37] P. Szańkowski, M. Trippenbach and Y. B. Band, “Decoherence of Atomic Clocks and Magnetometers due to Fluctuating Fields”, (to be published).
- [38] The moment generating functional is an alternative means of specifying the probability distributions of stochastic processes. The coefficients in the expansion of $M_X[\xi(t)]$ in powers of the trial function $\xi(t)$ define the moments of the process.